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On Regular Extensions of Contents and Measures

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INTRODUCTION

Let $\mathscr{A}$ be an algebra and $\mathscr{K} \subset \mathscr{L}$ lattices of subsets of a set X. Furthermore, let μ be a semifinite content on $\mathscr{A}$ such that $\mathscr{K}$ μ-approximates $\mathscr{A}$. Then our main result (3.4 and 3.6) states that μ can be extended to a semifinite, $\mathscr{L}$-regular content defined on an algebra containing both $\mathscr{A}$ and $\mathscr{L}$. If, in addition, μ is a measure and $\mathscr{L}$ is sequentially dominated by $\mathscr{A}$, then μ can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure defined on a σ-algebra containing both $\mathscr{A}$ and $\mathscr{L}$. Since, in the special case $\mathscr{K} \subset \mathscr{A}$, $\mathscr{K}$ μ-approximates $\mathscr{A}$ iff μ is $\mathscr{K}$-regular, our result is a common generalization of measure extension theorems due to Bachman and Sultan [5, 2.1], Lembcke [11, 3.1 and 4.5], Szeto [18, 2.2] and the author [3, 3.2 and 3.3(a)].

Our proceeding is based on the concept of a tight set function [2 and 3]. Some fundamental properties of tight set functions are compiled in Section 1. An essential tool for proving our main theorem is the fact that every supermodular set function defined on a lattice of sets admits a tight majorant. This result which is also important in cooperative game theory (see [6, 9, 16]) is proved in Section 2. A similar result concerning the existence of tight minorants for certain submodular set functions which has been proved in [1] is the basic tool for the derivation of another general measure extension theorem (3.12). As a consequence of this theorem, it is shown, among others, that every tight Baire measure on a [completely] Hausdorff space can be [uniquely] extended to a Radon measure (3.16).

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1. Definitions and Auxiliary Results

Let X be an arbitrary set. If A is a subset of X and f is a function defined on X, then $f \mid A$ denotes the restriction of f on A. A *paving* (in X) is a subset of $\mathscr{P}(X)$, the power set of X, which contains the empty set. A paving that is closed under finite [countable] intersections and finite unions is called a *lattice* [*δ-lattice*] of sets.

Let $\mathscr{C}$ be a paving in X. Then $\alpha(\mathscr{C})$ [$\sigma(\mathscr{C})$] denotes the [σ-]algebra generated by $\mathscr{C}$. Furthermore, $\mathscr{C}_\delta$ denotes the family of all countable intersections of sets from $\mathscr{C}$. $\mathscr{C}$ is said to be *semicompact* if every countable subfamily of $\mathscr{C}$ having the finite intersection property has nonvoid intersection. In the following let $\mathscr{K}$, $\mathscr{L}$ be lattices of subsets of X. $\mathscr{K}$ is said to be *sequentially dominated by $\mathscr{L}$* if whenever (K_n) is a sequence in $\mathscr{K}$ with $K_n \downarrow \varnothing$, there exists a sequence (L_n) in $\mathscr{L}$ such that $L_n \downarrow \varnothing$ and $K_n \subset L_n$ for all $n \in N$. We say *$\mathscr{L}$ separates $\mathscr{K}$* if whenever $K_1, K_2 \in \mathscr{K}$ with $K_1 \cap K_2 = \varnothing$, there are disjoint $\mathscr{L}$-sets L_1, L_2 such that $K_i \subset L_i$ for $i = 1, 2$.

$\mathscr{F}(\mathscr{K}) := \{F \in \mathscr{P}(X) \colon F \cap K \in \mathscr{K}$ for all $K \in \mathscr{K}\}$ denotes the lattice of "*local $\mathscr{K}$-sets*." Obviously $X \in \mathscr{F}(\mathscr{K})$ and $\mathscr{K} \subset \mathscr{F}(\mathscr{K}) \subset \mathscr{F}(\mathscr{K}_\delta)$. Moreover, we have $\mathscr{K} = \mathscr{F}(\mathscr{K})$ iff $X \in \mathscr{K}$.

Furthermore, we denote by $N(\mathscr{K})$ the family of all $[0, \infty]$-valued set functions defined on $\mathscr{K}$ and vanishing at $\varnothing$. If $\mathscr{K}$ is an algebra, then a finitely [countably] additive element of $N(\mathscr{K})$ is called a *content* [*measure*] on $\mathscr{K}$.

For a given $\lambda \in N(\mathscr{K})$ we define set functions λ_*, λ^* on $\mathscr{P}(X)$ by $\lambda_*(Q) := \sup\{\lambda(K) \colon K \in \mathscr{K}, K \subset Q\}$ and $\lambda^*(Q) := \inf\{\lambda(K) \colon K \in \mathscr{K}, K \supset Q\}$ (with $\inf \varnothing = \infty$). Moreover, we put $\|\lambda\| := \lambda_*(X)$.

Let $\mathscr{C}$ be a subpaving of $\mathscr{K}$. $\lambda \in N(\mathscr{K})$ is said to be

(a) *$\mathscr{C}$-regular* if $\lambda(K) = \sup\{\lambda(C) \colon C \in \mathscr{C}, C \subset K\}$ for $K \in \mathscr{K}$;

(b) *finite* if $\lambda(K) < \infty$ for all $K \in \mathscr{K}$;

(c) *semifinite* if $\lambda(K) = \infty$ implies $\lambda(K) = \sup\{\lambda(K_0) \colon K_0 \in \mathscr{K}, K_0 \subset K, \lambda(K_0) < \infty\}$;

(d) *supermodular* if $\lambda(K_1) + \lambda(K_2) \leqq \lambda(K_1 \cap K_2) + \lambda(K_1 \cup K_2)$ for $K_1, K_2 \in \mathscr{K}$;

(e) *modular* if $\lambda(K_1) + \lambda(K_2) = \lambda(K_1 \cap K_2) + \lambda(K_1 \cup K_2)$ for $K_1, K_2 \in \mathscr{K}$;

(f) *tight* if $\lambda(K_2) = \lambda(K_1) + \lambda_*(K_2 - K_1)$ for $K_1, K_2 \in \mathscr{K}$ with $K_1 \subset K_2$;

(g) *σ-smooth at $\varnothing$* if $\lim_n \lambda(K_n) = 0$ for every sequence (K_n) in $\mathscr{K}$ satisfying $K_n \downarrow \varnothing$ and $\inf_n \lambda(K_n) < \infty$.

For $\rho \in N(\mathscr{P}(X))$ denote by $\mathscr{M}(\rho) := \{A \in \mathscr{P}(X) \colon \rho(Q) = \rho(Q \cap A) + \rho(Q - A)$ for all $Q \subset X\}$ the family of so-called *ρ-measurable* subsets of X. It is well known (see [10, Lemma 1]) that $\mathscr{M}(\rho)$ is an algebra and ρ is a content on $\mathscr{M}(\rho)$. Furthermore, we need the following facts about tight set functions.

1.1. LEMMA [2, 2.1 and 2.2]. *Let $\lambda \in N(\mathscr{K})$ be tight. Then*

(a) *λ is monotone and modular;*

(b) *$\mathscr{F}(\mathscr{K}) \subset \mathscr{M}(\lambda_*)$;*

(c) *λ is σ-smooth at $\varnothing$ if $\mathscr{K}$ is semicompact.*

1.2. LEMMA. *Let $\lambda \in N(\mathscr{K})$ be tight with $\|\lambda\| < \infty$. Then we have $\mathscr{M}(\lambda_*) = \{A \in \mathscr{P}(X): \lambda_*(A) + \lambda_*(X - A) = \|\lambda\|\}$.*

Proof. For $A \in \mathscr{M}(\lambda_*)$, the equality

$$\lambda_*(A) + \lambda_*(X - A) = \|\lambda\| \tag{1.1}$$

trivially holds. Now assume that $A \in \mathscr{P}(X)$ satisfies (1.1). As λ_* is supermodular, we obtain, for any $Q \subset X$, $\lambda_*(Q) \geqq \lambda_*(Q \cap A) + \lambda_*(Q - A) \geqq \lambda_*(Q) + \lambda_*(A) - \lambda_*(Q \cup A) + \lambda_*(Q) + \lambda_*(X - A) - \lambda_*(Q \cup (X - A)) = \|\lambda\| + 2\lambda_*(Q) - \lambda_*(Q \cup A) - \lambda_*(Q \cup (X - A)) \geqq \|\lambda\| + 2\lambda_*(Q) - \lambda_*(X) - \lambda_*(Q) = \lambda_*(Q)$ which implies $A \in \mathscr{M}(\lambda_*)$. ∎

1.3. LEMMA. *Let $\lambda \in N(\mathscr{K})$ be tight, semifinite, and σ-smooth at $\varnothing$. Define a set function $\lambda^s \in N(\mathscr{P}(X))$ by*

$$\lambda^s(Q) := \sup\{\lambda^*(\bar{K}): \bar{K} \in \mathscr{K}_\delta, \bar{K} \subset Q \text{ and } \lambda^*(\bar{K}) < \infty\} \text{ for } Q \in \mathscr{P}(X). \tag{1.2}$$

Then we have

(a) *$\mathscr{M}(\lambda^s)$ is a σ-algebra containing both $\mathscr{F}(\mathscr{K}_\delta)$ and $\mathscr{M}(\lambda_*)$.*

(b) *$\lambda^s \mid \mathscr{M}(\lambda^s)$ is a semifinite, $\mathscr{K}_\delta$-regular measure which extends λ.*

(c) *$\lambda_* \mid \mathscr{M}(\lambda_*) = \lambda^s \mid \mathscr{M}(\lambda_*)$.*

(d) *If $\mathscr{K}$ is a δ-lattice, then $\lambda^s = \lambda_*$.*

Proof. See [2, 2.4–2.6], for a proof of (a), (b), and (d).

Ad (c): The inequality $\lambda_* \leqq \lambda^s$ is obvious. We first prove

$$\lambda_*(K \cap C) = \lambda^s(K \cap C) \qquad \text{for } C \in \mathscr{M}(\lambda_*) \text{ and } K \in \mathscr{K} \text{ with } \lambda(K) < \infty. \tag{1.3}$$

Let $C \in \mathscr{M}(\lambda_*)$ and $K \in \mathscr{K}$ with $\lambda(K) < \infty$ be given. By [2, 2.5], we then have $\lambda^s(K \cap C) + \lambda^s(K - C) \leqq \lambda^s(K) = \lambda(K) = \lambda_*(K \cap C) + \lambda_*(K - C) \leqq \lambda^s(K \cap C) + \lambda^s(K - C)$, hence $\lambda^s(K \cap C) + \lambda^s(K - C) = \lambda_*(K \cap C) + \lambda_*(K - C) = \lambda(K) < \infty$ which, together with $\lambda_* \leqslant \lambda^s$, implies (1.3).

Now let $C \in \mathscr{M}(\lambda_*)$ be given. Assume that $\lambda_*(C) < \lambda^s(C)$. As λ^s is $\mathscr{K}_\delta$-regular and semifinite, there is a $\mathscr{K}_\delta$-set $\bar{K}$ with $\bar{K} \subset C$ and $\lambda_*(C) < \lambda^s(\bar{K}) = \lambda^*(\bar{K}) < \infty$. Choose a decreasing sequence (K_n) in $\mathscr{K}$ with $\bar{K} = \bigcap_n K_n$ and $\lambda(K_n) < \infty$ for all n. Then $\bar{K} = \bigcap_n (K_n \cap C)$ and so, by (1.3), $\lambda_*(C) < \lambda^s(\bar{K}) = \inf_n \lambda^s(K_n \cap C) = \inf_n \lambda_*(K_n \cap C) \leqq \lambda_*(C)$. This contradiction proves our claim. ∎

Finally, we use the following topological notation. If X is a topological space, then any set of the form $\{f = 0\}$, where f is a continuous real-valued function on X is called a *zero-set* in X. Complements of zero-sets are called *cozero-sets*. We denote by $\mathscr{Z}(X)$, $\mathscr{U}(X)$, $\mathscr{F}(X)$, $\mathscr{G}(X)$, and $\mathscr{K}(X)$ the family of all zero-, cozero-, closed, open, and compact sets in X, respectively. $\mathscr{B}_0(X) := \sigma(\mathscr{Z}(X))$ $[\mathscr{B}(X) := \sigma(\mathscr{F}(X))]$ denotes the *Baire* [*Borel*] σ-algebra in X. Any measure defined on $\mathscr{B}_0(X)$ $[\mathscr{B}(X)]$ is called a *Baire* [*Borel*] *measure* on X.

2. Tight Majorants of Supermodular Set Functions

Throughout this section $\mathscr{K}$ denotes a lattice of subsets of an arbitrary set X. We introduce a partial order in the set $N(\mathscr{K})$ by defining $\lambda_1 \leqq \lambda_2$ iff $\lambda_1(K) \leqq \lambda_2(K)$ for all $K \in \mathscr{K}$.

2.1. Lemma. *For $a \in [0, \infty)$ let $\Lambda^a := \{\lambda \in N(\mathscr{K})$: λ supermodular and $\|\lambda\| = a\}$. Then*

(a) *every nonempty linearly ordered subset of Λ^a has an upper bound in Λ^a and*

(b) *every maximal element of Λ^a is tight.*

Proof. (a) Let $\Lambda_0 := \{\lambda_i : i \in I\}$ be a nonvoid linearly ordered subset of Λ^a. Put $\lambda(K) := \sup\{\lambda_i(K) : i \in I\}$ for $K \in \mathscr{K}$. It is clear that $\lambda \in N(\mathscr{K})$ and $\|\lambda\| = a$. To prove the supermodularity of λ, let $K_1, K_2 \in \mathscr{K}$ and $\varepsilon > 0$ be given. Choose i_1, i_2 in I such that $\lambda(K_j) < \lambda_{i_j}(K_j) + \varepsilon/2$ for $j = 1, 2$. Suppose $\lambda_{i_1} \leqq \lambda_{i_2}$. Then $\lambda(K_1) + \lambda(K_2) < \lambda_{i_1}(K_1) + \lambda_{i_2}(K_2) + \varepsilon \leqslant \lambda_{i_2}(K_1) + \lambda_{i_2}(K_2) + \varepsilon \leqq \lambda_{i_2}(K_1 \cap K_2) + \lambda_{i_2}(K_1 \cup K_2) + \varepsilon \leqq \lambda(K_1 \cap K_2) + \lambda(K_1 \cup K_2) + \varepsilon$.It follows that λ is supermodular. Hence $\lambda \in \Lambda^a$ is an upper bound for Λ_0.

(b) Let λ be a maximal element of Λ^a. For any subset C of X, define a set function λ_C on $\mathscr{K}$ by $\lambda_C(K) := \lambda_*(K \cap C) + \lambda_*(K \cup C) - \lambda_*(C)$, $K \in \mathscr{K}$. We now prove that

$$\lambda_C \in \Lambda^a. \tag{2.1}$$

Trivially we have $\lambda_C \in N(\mathscr{K})$. In order to prove the supermodularity of λ_C, it suffices to show that, for any $\mathscr{K}$-sets K_1 and K_2, we have $\lambda_*(K_1 \cap C) +$

$\lambda_*(K_1 \cup C) + \lambda_*(K_2 \cap C) + \lambda_*(K_2 \cup C) \leqq \lambda_*(K_1 \cap K_2 \cap C) + \lambda_*((K_1 \cap K_2) \cup C) + \lambda_*((K_1 \cup K_2) \cap C) + \lambda_*(K_1 \cup K_2 \cup C)$. But this follows immediately from the two inequalities $\lambda_*(K_1 \cap C) + \lambda_*(K_2 \cap C) \leqq \lambda_*(K_1 \cap K_2 \cap C) + \lambda_*((K_1 \cup K_2) \cap C)$ and $\lambda_*(K_1 \cup C) + \lambda_*(K_2 \cup C) \leqq \lambda_*((K_1 \cap K_2) \cup C) + \lambda_*(K_1 \cup K_2 \cup C)$. Furthermore, we have

$$\|\lambda_C\| = a. \tag{2.2}$$

It is clear that $\lambda_C \geqq \lambda \in \Lambda^a$ implies $\|\lambda_C\| \geqq a$. Assuming $\|\lambda_C\| > a$, we have $\lambda_*(K \cap C) + \lambda_*(K \cup C) - \lambda_*(C) > a$ for some $\mathscr{K}$-set K and hence $\lambda_*(C) + a < \lambda_*(K \cap C) + \lambda_*(K \cup C) \leqq \lambda_*(C) + a$. This contradiction proves (2.2) and hence (2.1).

As we have $\lambda_C \geqq \lambda$, the maximality of λ implies $\lambda = \lambda_C$ for every subset C of X. In particular, for $\mathscr{K}$-sets K_1, K_2 with $K_1 \subset K_2$, we obtain $\lambda(K_1) = \lambda_{K_2 - K_1}(K_1) = \lambda(K_2) - \lambda_*(K_2 - K_1)$. Hence λ is tight. ∎

We can now prove the main result of this section. It is an essential tool for the derivation of a general measure extension theorem in the next section.

2.2. THEOREM. *For any supermodular* $\gamma \in N(\mathscr{K})$ *there exists a tight* $\lambda \in N(\mathscr{K})$ *such that* $\gamma \leqq \lambda$ *and* $\|\gamma\| = \|\lambda\|$.

Proof. If $\|\gamma\| = \infty$, then the tight set function λ defined by $\lambda(K) = \infty$ for all nonvoid $\mathscr{K}$-sets K is appropriate. If $a := \|\gamma\| < \infty$, then, by 2.1(a) and Zorn's lemma, there is a maximal element $\lambda \in \Lambda^a$ with $\lambda \geqq \gamma$. Thus our claim follows from 2.1(b). ∎

2.3. *Remarks.* (a) In the special case of a modular, monotone $\gamma \in N(\mathscr{K})$ and $X \in \mathscr{K}$, 2.2 was proved independently by Pachl [14] and Szeto [18].

(b) If the supermodular set function γ is $\{0, 1, ..., n\}$-valued (for some $n \in N$), then a straightforward modification of the proof of 2.1 shows that the tight majorant λ in 2.2 can be chosen $\{0, 1, ..., n\}$-valued, too.

(c) If $\mathscr{K}$ is an algebra, then the tight set functions on $\mathscr{K}$ are exactly the contents on $\mathscr{K}$. In this case, 2.2 is a well-known result ("core theorem") in cooperative game theory. The reader interested in the game theoretic aspects of nonnegative set functions is referred to [6, 9, 16].

(d) If, under the assumptions of 2.2, γ is finite but $\|\gamma\| = \infty$, then one cannot conclude that there is a tight majorant λ which is semifinite or even finite: Let $\mathscr{K}$ be the family of all finite subsets of the real line and put $\gamma(K) := (\operatorname{card} K)^2$ for $K \in \mathscr{K}$. Then γ is supermodular and finite. An analysis of example (5.6) in [4] reveals that there is no modular $\lambda \in N(\mathscr{K})$

satisfying $\gamma \leqq \lambda$ and $\lambda(\{x\}) < \infty$ for all real numbers x. In particular, there does not exist any semifinite tight $\lambda \in N(\mathscr{K})$ with $\gamma \leqq \lambda$.

(e) If $\gamma \in N(\mathscr{K})$ is supermodular and σ-smooth at $\varnothing$, then one cannot infer from 2.2 that there is a tight majorant λ which is also σ-smooth at $\varnothing$: Let X be any set with cardinality $\aleph_1$, $\mathscr{K}$ the power set of X, and $\mathscr{D}$ the family of all subsets of X having a countable complement. For any $K \in \mathscr{K}$, put $\gamma(K) = 1$ or 0 according as $K \in \mathscr{D}$ or $K \notin \mathscr{D}$. Then $\gamma \in N(\mathscr{K})$ is supermodular and σ-smooth at $\varnothing$. On the other hand, there is no probability measure λ on $\mathscr{K}$ with $\gamma \leqq \lambda$, since otherwise we would have $\lambda(\{x\}) = 0$ for all $x \in X$, which is a contradiction to Ulam's theorem [13, Theorem 5.6].

3. Regular Extensions of Contents and Measures

Throughout this section X denotes an arbitrary set, whereas $\mathscr{K}$, $\mathscr{L}$ are lattices of subsets of X and $\mathscr{A}$ is an algebra in X. We begin with the following uniqueness statement.

3.1. Proposition. *Assume that $\mathscr{A}$ separates $\mathscr{K}$.*

(a) *If $\mathscr{A}'$ is an algebra containing $\mathscr{A}$ and $\mathscr{K}$, then every finite content on $\mathscr{A}$ has at most one extension to a $\mathscr{K}$-regular content on $\mathscr{A}'$.*

(b) *If $\mathscr{A}'$ is a σ-algebra containing $\mathscr{A}$ and $\mathscr{K}$, then every σ-finite measure on $\mathscr{A}$ has at most one extension to a $\mathscr{K}$-regular measure on $\mathscr{A}'$.*

Proof. (a) Let μ be a finite content on $\mathscr{A}$. Suppose that μ_1, μ_2 are $\mathscr{K}$-regular contents on $\mathscr{A}'$ that extend μ. In order to prove $\mu_1 = \mu_2$ it suffices to show $\mu_1(X-K) = \mu_2(X-K)$ for all $K \in \mathscr{K}$. Assume that $\mu_1(X-K_0) \neq \mu_2(X-K_0)$, say $\mu_1(X-K_0) < \mu_2(X-K_0)$ for some $\mathscr{K}$-set K_0. Then there exists a $\mathscr{K}$-set K with $K \subset X-K_0$ and $\mu_1(X-K_0) < \mu_2(K)$. Since $\mathscr{A}$ separates $\mathscr{K}$, there are disjoint $\mathscr{A}$-sets A and A_0 such that $K \subset A$ and $K_0 \subset A_0$. This implies $\mu(A) \leqq \mu(X-A_0) \leqq \mu_1(X-K_0) < \mu_2(K) \leqq \mu(A)$ which is impossible.

(b) Let μ be a σ-finite measure on $\mathscr{A}$. Then there is a disjoint sequence (X_n) in $\mathscr{A}$ such that $X = \bigcup_n X_n$ and $\mu(X_n) < \infty$ for all $n \in N$. Let v and τ be $\mathscr{K}$-regular measures on $\mathscr{A}'$ which extend μ. For $n \in N$, $A \in \mathscr{A}$, and $A' \in \mathscr{A}'$, define $\mu_n(A) := \mu(A \cap X_n)$, $v_n(A') := v(A' \cap X_n)$, and $\tau_n(A') := \tau(A' \cap X_n)$. μ_n is a finite measure on $\mathscr{A}$, whereas v_n and τ_n are finite, $\mathscr{K}$-regular measures on $\mathscr{A}'$. Since v_n and τ_n extend μ_n, we obtain $v_n = \tau_n$, $n \in N$, by (a), and so $v = \tau$. ∎

The example 3.5 in [3] shows that 3.1 does not remain valid for semifinite measures. The following extension result is an application of 2.2.

3.2. PROPOSITION. *Let $\gamma \in N(\mathscr{K})$ be supermodular with $\|\gamma\| < \infty$.*

(a) *If $\gamma_* \mid \mathscr{A}$ is a content, then $\gamma_* \mid \mathscr{A}$ can be extended to a $\mathscr{K}$-regular content on $\alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{K}))$.*

(b) *If $\mathscr{K}$ is sequentially dominated by $\mathscr{A}$ and $\gamma_* \mid \mathscr{A}$ is a measure, then $\gamma_* \mid \mathscr{A}$ can be extended to a $\mathscr{K}_\delta$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$.*

Proof. By 2.2, there is a tight $\lambda \in N(\mathscr{K})$ such that $\gamma \leqq \lambda$ and $\|\gamma\| = \|\lambda\|$. Then, for any $A \in \mathscr{A}$, we have $\gamma_*(A) \leqq \lambda_*(A) \leqq \|\lambda\| - \lambda_*(X - A) \leqq \|\gamma\| - \gamma_*(X - A) = \gamma_*(A)$, hence $\gamma_*(A) = \lambda_*(A) = \|\lambda\| - \lambda_*(X - A)$ for all $A \in \mathscr{A}$ which implies $\mathscr{A} \subset \mathscr{M}(\lambda_*)$ by 1.2. On the other hand, we have $\mathscr{F}(\mathscr{K}) \subset \mathscr{M}(\lambda_*)$ by 1.1(b). Thus $\mathscr{A}' := \alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{K})) \subset \mathscr{M}(\lambda_*)$ which implies that $\lambda_* \mid \mathscr{A}'$ is a $\mathscr{K}$-regular content that extends $\gamma_* \mid \mathscr{A}$. This proves (a).

Next we show that, under the additional assumptions of (b), λ is σ-smooth at $\varnothing$. For this purpose, let $(K_n) \subset \mathscr{K}$ with $K_n \downarrow \varnothing$. Then there is a sequence (A_n) in $\mathscr{A}$ such that $A_n \downarrow \varnothing$ and $K_n \subset A_n$ for all n. This implies $\lambda(K_n) \leqslant \lambda_*(A_n) = \gamma_*(A_n) \to 0$ for $n \to \infty$. Hence λ is σ-smooth at $\varnothing$. Put $\hat{\mathscr{A}} := \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$ and define λ^s by (1.2). Then we infer from 1.3 that $\hat{\mathscr{A}} \subset \mathscr{M}(\lambda^s)$ and $\lambda^s \mid \hat{\mathscr{A}}$ is a $\mathscr{K}_\delta$-regular measure which extends $\gamma_* \mid \mathscr{A}$. ∎

Let μ be a content defined on the algebra $\mathscr{A}$. Following the terminology of [15] we say that $\mathscr{K}$ *μ-approximates* $\mathscr{A}$ if $\mu(A) = \sup\{\mu_*(K): K \in \mathscr{K}, K \subset A\}$ for all $A \in \mathscr{A}$.

3.3. *Remarks.* (a) If $\mathscr{K} \subset \mathscr{A}$, then $\mathscr{K}$ μ-approximates $\mathscr{A}$ iff μ is $\mathscr{K}$-regular.

(b) If μ is finite, then $\mathscr{K}$ μ-approximates $\mathscr{A}$ iff for every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, there exist sets $A_0 \in \mathscr{A}$ and $K \in \mathscr{K}$ such that $A_0 \subset K \subset A$ and $\mu(A - A_0) < \varepsilon$. The latter condition goes back to Marczewski [12].

(c) If $\mathscr{K}$ μ-approximates $\mathscr{A}$, where $\mathscr{K}$ is semicompact and μ is semifinite, then μ is σ-additive.

Proof. According to [2, 4.2(a)], it suffices to show that μ is σ-smooth at $\varnothing$. But this can be done by standard arguments (see [12, (i), p. 118]). ∎

We can now prove the following basic extension theorem.

3.4. THEOREM. *Let μ be a semifinite content on $\mathscr{A}$ such that $\mathscr{K}$ μ-approximates $\mathscr{A}$. Furthermore assume that $\mathscr{K} \subset \mathscr{L}$.*

(a) *Then μ can be extended to a semifinite, $\mathscr{L}$-regular content on $\alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{L}))$. The extension is unique on every algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{L}))$ if μ is finite and $\mathscr{A}$ separates $\mathscr{L}$.*

(b) *If, in addition, μ is σ-additive and $\mathscr{L}$ is sequentially dominated by $\mathscr{A}$, then μ can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{L}_\delta))$. The extension is unique on every σ-algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{L}_\delta))$ if μ is σ-finite and $\mathscr{A}$ separates $\mathscr{L}_\delta$.*

Proof. The uniqueness assertions follow directly from 3.1. Since $\mathscr{K}$ μ-approximates $\mathscr{A}$, so does $\mathscr{L}$. Thus in order to prove the existence statements it suffices to consider the special case $\mathscr{K} = \mathscr{L}$. We first assume $\mu(X) < \infty$. Putting $\gamma(K) := \mu_*(K)$ for $K \in \mathscr{K}$, we have $\mu = \gamma_* \mid \mathscr{A}$. As $\gamma \in N(\mathscr{K})$ is supermodular with $\|\gamma\| = \mu(X) < \infty$, our claim follows from 3.2. Now assume that μ is semifinite. We only deal with the case where μ is a semifinite measure on $\mathscr{A}$ (and $\mathscr{K}$ is sequentially dominated by $\mathscr{A}$). The finitely additive case which is still simpler can be treated in an analogous way.

Let $\mathscr{M}$ denote the family of all $\mathscr{K}_\delta$-regular measures ν on $\mathscr{A}' := \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$ satisfying $\nu \mid \mathscr{A} \leqq \mu$. Obviously $0 \in \mathscr{M}$. We first prove that $\mathscr{M}$ is inductively ordered. For this purpose let $\mathscr{M}_0 \subset \mathscr{M}$ be linearly ordered and put $\nu_0 := \sup \mathscr{M}_0$, i.e., $\nu_0(A') = \sup\{\sum_n \nu_n(A_n)$: $(A_n) \subset \mathscr{A}'$ is a decomposition of A' and $(\nu_n) \subset \mathscr{M}_0\}$, $A' \in \mathscr{A}'$. It is well known that ν_0 is a measure on $\mathscr{A}'$ satisfying $\nu_0 \geqq \nu$ for all $\nu \in \mathscr{M}_0$. Furthermore,

$$\nu_0 \text{ is } \mathscr{K}_\delta\text{-regular.} \tag{3.1}$$

Let $A' \in \mathscr{A}'$ and $a < \nu_0(A')$. Then there are measures $\nu_1, \nu_2, \ldots$ in $\mathscr{M}_0$ and pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathscr{A}'$ such that $A' = \bigcup_n A_n$ and $a < \sum_n \nu_n(A_n)$. Then $a < \sum_{n=1}^k \nu_n(A_n)$ for some index $k \in N$. Choose a number $k_0 \in \{1, \ldots, k\}$ such that $\nu_{k_0} \geqq \nu_n$ for $n = 1, \ldots, k$. Then $a < \sum_{n=1}^k \nu_n(A_n) \leqq \sum_{n=1}^k \nu_{k_0}(A_n) = \nu_{k_0}(\bigcup_{n=1}^k A_n) \leqq \nu_{k_0}(A')$. As ν_{k_0} is $\mathscr{K}_\delta$-regular, there is a $\mathscr{K}_\delta$-set $\bar{K} \subset A'$ such that $a < \nu_{k_0}(\bar{K})$ and hence $a < \nu_0(\bar{K})$. This proves (3.1).

Now let $A \in \mathscr{A}$ be given. Furthermore, let $\nu_1, \nu_2, \ldots \in \mathscr{M}_0$ and $A_1, A_2, \ldots$ pairwise disjoint $\mathscr{A}'$-sets with union A. For any $k \in N$, one can find an index $k_0 \in \{1, \ldots, k\}$ such that $\nu_{k_0} \geqq \nu_n$ for $n = 1, \ldots, k$. This implies $\sum_{n=1}^k \nu_n(A_n) \leqq \sum_{n=1}^k \nu_{k_0}(A_n) = \nu_{k_0}(\bigcup_{n=1}^k A_n) \leqq \nu_{k_0}(A) \leqq \mu(A)$. As $k \in N$ was arbitrary, we obtain $\sum_n \nu_n(A_n) \leqq \mu(A)$ and hence $\nu_0(A) \leqq \mu(A)$. Thus we have $\nu_0 \mid \mathscr{A} \leqq \mu$ which together with (3.1) implies $\nu_0 \in \mathscr{M}$. Hence $\mathscr{M}$ is inductively ordered. By Zorn's lemma, there is a maximal element $\tilde{\mu}$ in $\mathscr{M}$.

Now let $A_0 \in \mathscr{A}$ with $\mu(A_0) < \infty$ be given. Put $\nu(A) := \mu(A \cap A_0) - \tilde{\mu}(A \cap A_0)$ for $A \in \mathscr{A}$. Then ν is a finite measure on $\mathscr{A}$. Furthermore,

$$\mathscr{K} \ \nu\text{-approximates } \mathscr{A}. \tag{3.2}$$

In order to prove (3.2), let $A \in \mathscr{A}$ and $\varepsilon > 0$ be given. There exist sets $K \in \mathscr{K}$ and $\tilde{A} \in \mathscr{A}$ such that $\tilde{A} \subset K \subset A \cap A_0$ and $\mu(A \cap A_0) < \mu(\tilde{A}) + \varepsilon$. Then $\nu(A) - \nu(\tilde{A}) = \mu(A \cap A_0) - \mu(\tilde{A}) - (\tilde{\mu}(A \cap A_0) - \tilde{\mu}(\tilde{A})) \leqq \mu(A \cap A_0) - \mu(\tilde{A}) < \varepsilon$. Thus (3.2) holds.

By the "finite case," ν can be extended to a $\mathcal{K}_\delta$-regular measure $\tilde{\nu}$ on $\mathcal{A}'$. Then $\tau := \tilde{\mu} + \tilde{\nu}$ is a $\mathcal{K}_\delta$-regular measure on $\mathcal{A}'$, and for any $\mathcal{A}$-set A we have $\tau(A) = \tau(A \cap A_0) + \tau(A - A_0) = \tilde{\mu}(A \cap A_0) + \tilde{\nu}(A \cap A_0) + \tilde{\mu}(A - A_0) + \tilde{\nu}(A - A_0) = \tilde{\mu}(A \cap A_0) + \nu(A \cap A_0) + \tilde{\mu}(A - A_0) + \nu(A - A_0) = \mu(A \cap A_0) + \tilde{\mu}(A - A_0) \leqq \mu(A \cap A_0) + \mu(A - A_0) = \mu(A)$. Thus we have $\tau \in \mathcal{M}$. Since $\tilde{\mu} \leqq \tau$, the maximality of $\tilde{\mu}$ implies $\tilde{\mu} = \tau$. In particular, we have $\infty > \mu(A_0) \geqq \tilde{\mu}(A_0) = \tau(A_0) = \tilde{\mu}(A_0) + \tilde{\nu}(A_0)$, hence $\tilde{\nu}(A_0) = 0$ and so $\mu(A_0) = \tilde{\mu}(A_0)$. As the set A_0 was arbitrary, we thus have proved

$$\mu(A) = \tilde{\mu}(A) \qquad \text{for all } A \in \mathcal{A} \text{ with } \mu(A) < \infty. \tag{3.3}$$

For any $A \in \mathcal{A}'$, put $\tilde{\mu}_1(A) := \sup\{\tilde{\mu}(B) \colon B \in \mathcal{A}',\ B \subset A \text{ and } \tilde{\mu}(B) < \infty\}$. Then $\tilde{\mu}_1$ is a semifinite measure on $\mathcal{A}'$. In addition,

$$\tilde{\mu}_1 \text{ is } \mathcal{K}_\delta\text{-regular.} \tag{3.4}$$

Let $A \in \mathcal{A}'$ and $a < \tilde{\mu}_1(A)$. There exists a set $B \in \mathcal{A}'$ with $B \subset A$ and $a < \tilde{\mu}(B) < \infty$. As $\tilde{\mu}$ is $\mathcal{K}_\delta$-regular, we can find a $\mathcal{K}_\delta$-set $\bar{K}$ such that $\bar{K} \subset B$ and $a < \tilde{\mu}(\bar{K}) < \infty$. Then $\tilde{\mu}(\bar{K}) = \tilde{\mu}_1(\bar{K})$ which proves (3.4).

For any $A_0 \in \mathcal{A}$ with $\mu(A_0) < \infty$, we have $\infty > \mu(A_0) = \tilde{\mu}(A_0) = \tilde{\mu}_1(A_0)$ by (3.3). This implies, for any $\mathcal{A}$-set A, $\tilde{\mu}_1(A) \leqq \tilde{\mu}(A) \leqq \mu(A) = \sup\{\mu(A_0) \colon A_0 \in \mathcal{A},\ A_0 \subset A,\ \mu(A_0) < \infty\} = \sup\{\tilde{\mu}_1(A_0) \colon A_0 \in \mathcal{A},\ A_0 \subset A,\ \mu(A_0) < \infty\} \leqq \tilde{\mu}_1(A)$. Thus $\tilde{\mu}_1$ is a semifinite, $\mathcal{K}_\delta$-regular measure extension of μ. ∎

3.5. PROPOSITION. *Assume that $\mathcal{K} \subset \mathcal{L}$ where $\mathcal{L}$ is sequentially dominated by $\mathcal{A}$.*

(a) *If μ is a (not necessarily semifinite) measure on $\mathcal{A}$ such that $\mathcal{K}$ μ-approximates $\mathcal{A}$, then μ can be extended to a measure on $\sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{L}_\delta))$.*

(b) *If $\mathcal{K}$ μ-approximates $\mathcal{A}$ for every finite measure μ on $\mathcal{A}$, then every measure on $\mathcal{A}$ can be extended to a measure on $\sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{L}_\delta))$.*

Proof. Let μ be an arbitrary measure on $\mathcal{A}$. For any $A \in \mathcal{A}$, put $\nu(A) := \sup\{\mu(A_0) \colon A_0 \in \mathcal{A},\ A_0 \subset A \text{ and } \mu(A_0) < \infty\}$ and $\tau(A) := \sup\{\mu(B) - \nu(B) \colon B \in \mathcal{A},\ B \subset A \text{ and } \nu(B) < \infty\}$. It is easy to verify that

$$\mu = \nu + \tau, \qquad \text{where } \nu\ [\tau] \text{ is a semifinite } [\{0, \infty\}\text{-valued] measure on } \mathcal{A}. \tag{3.5}$$

In particular, τ can be extended to a measure τ_1 on $\sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{L}_\delta))$ (see [3, 1.3]).

Ad (a). Let μ be a measure on $\mathscr{A}$ such that $\mathscr{K}$ μ-approximates $\mathscr{A}$. Decompose μ according to (3.5). Then it is easy to see that $\mathscr{K}$ ν-approximates $\mathscr{A}$. Thus we infer from 3.4(b) that ν can be extended to a measure ν_1 on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{L}_\delta))$. Hence $\nu_1 + \tau_1$ is the desired measure extension of μ.

Ad (b). Assume that $\mathscr{K}$ μ-approximates $\mathscr{A}$ for every finite measure μ on $\mathscr{A}$. This implies that $\mathscr{K}$ μ-approximates $\mathscr{A}$ for every semifinite measure μ on $\mathscr{A}$. Using the decomposition (3.5), our claim now follows from 3.4(b). ∎

Under the additional assumption $\mathscr{K} \subset \mathscr{A}$, Theorem 3.4 can be strengthened in the following way.

3.6. THEOREM. *Assume that $\mathscr{K} \subset \mathscr{A} \cap \mathscr{L}$. Then we have:*

(a) *Every semifinite, $\mathscr{K}$-regular content μ on $\mathscr{A}$ can be extended to a semifinite, $\mathscr{L}$-regular content on $\alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{K}) \cup \mathscr{F}(\mathscr{L}))$. The extension is unique on every algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{K}) \cup \mathscr{F}(\mathscr{L}))$ if μ is finite and $\alpha(\mathscr{A} \cup \mathscr{F}(\mathscr{K}))$ separates $\mathscr{L}$.*

(b) *If, in addition, $\mathscr{L}$ is sequentially dominated by $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$, then every semifinite, $\mathscr{K}$-regular measure μ on $\mathscr{A}$ can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$. The extension is unique on every σ-algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$ if μ is σ-finite and $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$ separates $\mathscr{L}_\delta$.*

Proof. We only prove (b), since the (simpler) proof of (a) can be performed in the same way. Define $\mathscr{C} := \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$ and $\mathscr{D} := \sigma(\mathscr{C} \cup \mathscr{F}(\mathscr{L}_\delta)) = \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$. Let μ be a semifinite, $\mathscr{K}$-regular measure on $\mathscr{A}$. Set $\lambda := \mu \mid \mathscr{K}$. Then $\lambda \in N(\mathscr{K})$ is tight, semifinite, and σ-smooth at $\varnothing$.

Now we let $A \in \mathscr{A}$ be given. For any $K \in \mathscr{K}$, we have $\lambda(K) = \mu(K) = \mu(K \cap A) + \mu(K - A) = \lambda_*(K \cap A) + \lambda_*(K - A)$ which implies $A \in \mathscr{M}(\lambda_*)$ by [2, 1.1d]. Thus we have $\mathscr{A} \subset \mathscr{M}(\lambda_*)$, and from 1.3(a)–(c) we obtain $\mathscr{C} \subset \mathscr{M}(\lambda^s)$, where $\nu := \lambda^s \mid \mathscr{C}$ is a semifinite, $\mathscr{K}_\delta$-regular measure which extends $\mu = \lambda_* \mid \mathscr{A}$. Since $\mathscr{L}$ is sequentially dominated by $\mathscr{C}$, so is $\mathscr{L}_\delta$. In addition, we have $\mathscr{K}_\delta \subset \mathscr{L}_\delta$. Thus we infer from 3.4(b) that ν can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure on $\mathscr{D}$.

In order to prove the uniqueness statement, let $\mathscr{A}'$ be any σ-algebra satisfying $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \mathscr{D}$. Assume that $\mathscr{C}$ separates $\mathscr{L}_\delta$ and that μ is finite. (The σ-finite case can be reduced to the finite case as in the proof of 3.1(b).) Let μ_1, μ_2 be $\mathscr{L}_\delta$-regular measures on $\mathscr{A}'$ that extend μ. Put $\rho := \mu_1 \mid \mathscr{L}_\delta$ and $\tau := \mu_2 \mid \mathscr{L}_\delta$. ρ and τ are finite, tight set functions on $\mathscr{L}_\delta$ being σ-smooth at $\varnothing$.

For any $Q \subset X$, we have $\lambda^s(Q) = \sup\{\lambda^*(\bar{K}): \bar{K} \in \mathscr{K}_\delta,\ \bar{K} \subset Q\} \leqq \sup\{\mu_1(\bar{K}): \bar{K} \in \mathscr{K}_\delta,\ \bar{K} \subset Q\} \leqq \sup\{\mu_1(\bar{L}):\ \bar{L} \in \mathscr{L}_\delta,\ \bar{L} \subset Q\} = \rho_*(Q)$, and $\lambda^s(X) = \mu(X) = \mu_1(X) = \rho_*(X)$. Thus we have

$$\lambda^s \leqslant \rho_* \qquad \text{and} \qquad \lambda^s(X) = \rho_*(X). \tag{3.6}$$

Furthermore, we have

$$\mathscr{F}(\mathscr{K}_\delta) \subset \mathscr{M}(\rho_*). \tag{3.7}$$

If $F \in \mathscr{F}(\mathscr{K}_\delta)$, then $F \in \mathscr{M}(\lambda^s)$ by 1.3(a), hence $\rho_*(F) + \rho_*(X-F) \leqslant \rho_*(X) = \lambda^s(X) = \lambda^s(F) + \lambda^s(X-F) \leqslant \rho_*(F) + \rho_*(X-F)$ by (3.6). Therefore we have $\rho_*(F) + \rho_*(X-F) = \rho_*(X)$ and so $F \in \mathscr{M}(\rho_*)$ by 1.2. This proves (3.7).

Moreover, we have $\mathscr{F}(\mathscr{L}_\delta) \subset \mathscr{M}(\rho_*)$ and $\mathscr{A} \subset \mathscr{M}(\rho_*)$ by 1.1 and 1.2, respectively. Since $\mathscr{M}(\rho_*)$ is a σ-algebra (see 1.3(a) and (d)), we obtain $\mathscr{D} \subset \mathscr{M}(\rho_*)$. As λ^s and ρ_* are finite measures on $\mathscr{C}$, (3.6) implies $\lambda^s | \mathscr{C} = \rho_* | \mathscr{C}$. In the same way, we can deduce $\lambda^s | \mathscr{C} = \tau_* | \mathscr{C}$ and $\mathscr{D} \subset \mathscr{M}(\tau_*)$. Thus $\rho_* | \mathscr{D}$ and $\tau_* | \mathscr{D}$ are finite, $\mathscr{L}_\delta$-regular measures (see 1.3) which coincide on $\mathscr{C}$. Hence 3.1(b) implies $\rho_* | \mathscr{D} = \tau_* | \mathscr{D}$, in particular, $\mu_1 = \mu_2$. ∎

3.7. Corollary. *Assume that $\mathscr{K} \subset \mathscr{L}$, where $\mathscr{L}$ is a semicompact lattice. Furthermore, let μ be a semifinite content on $\mathscr{A}$ such that $\mathscr{K}$ μ-approximates $\mathscr{A}$.*

(a) *Then μ can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{L}_\delta))$. The extension is unique on every σ-algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{L}_\delta))$ if μ is σ-finite and $\mathscr{A}$ separates $\mathscr{L}_\delta$.*

(b) *If, in addition, $\mathscr{K} \subset \mathscr{A}$ (i.e., μ is $\mathscr{K}$ -regular), then μ can be extended to a semifinite, $\mathscr{L}_\delta$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$. The extension is unique on every σ-algebra $\mathscr{A}'$ with $\mathscr{A} \cup \mathscr{L} \subset \mathscr{A}' \subset \sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$ if μ is σ-finite and $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$ separates $\mathscr{L}_\delta$.*

Proof. Since $\mathscr{L}$ is semicompact, so is $\mathscr{K}$. Thus μ is σ-additive by 3.3(c). On the other hand, the semicompactness of $\mathscr{L}$ implies that $\mathscr{L}$ is sequentially dominated by $\mathscr{A}$. Now the assertion follows from 3.4(b) and 3.6(b), respectively. ∎

Theorem 3.6 is a common generalization of [3, 3.2 and 3.3(a); 5, 2.1; and 18, 2.2]. Since any $\mathscr{K}$-regular content [measure] on a ring [σ-ring] $\mathscr{R}$ can be uniquely extended to a $\mathscr{K}$-regular content [measure] on $\alpha(\mathscr{R})$ [$\sigma(\mathscr{R})$], we also obtain from 3.6(a) [3.7(b)] Satz 3.1 [Satz 4.5] of [11]. The following result is an immediate consequence of 3.7(b).

3.8. Corollary. *Assume that $\mathscr{K} \subset \mathscr{L} \subset \mathscr{F}(\mathscr{K}_\delta)$ where $\mathscr{K}$ is a semicom-*

pact lattice. If $\mathscr{A} \subset \sigma(\mathscr{L})$, then every semifinite, $\mathscr{K} \cap \mathscr{A}$-regular content on $\mathscr{A}$ can be extended to a semifinite, $\mathscr{K}_\delta$-regular measure on $\sigma(\mathscr{L})$.

Corollary 3.8 is a generalization of Henry's extension theorem (see [17, Theorem 16, p. 51]) which one obtains from 3.8 if $\mathscr{K}$, $\mathscr{L}$ are the lattices of compact, respectively closed, subsets of a Hausdorff space X.

The following proposition can be derived from 3.6(b) in the same way as 3.5 has been deduced from 3.4(b).

3.9. PROPOSITION. *Assume that $\mathscr{K} \subset \mathscr{A} \cap \mathscr{L}$, where $\mathscr{L}$ is sequentially dominated by $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta))$.*

(a) *If μ is a (not necessarily semifinite) $\mathscr{K}$-regular measure on $\mathscr{A}$, then μ can be extended to a measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$.*

(b) *If every finite measure on $\mathscr{A}$ is $\mathscr{K}$-regular, then every measure on $\mathscr{A}$ can be extended to a measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}_\delta) \cup \mathscr{F}(\mathscr{L}_\delta))$.*

Further applications of 3.6 to extension problems in abstract and topological measure theory can be found in [3, Sect. 3]. The following two simple examples show that the assumption $\mathscr{K} \subset \mathscr{A}$ is essential for the validity of 3.6.

3.10. EXAMPLES. (a) Let X be a set with cardinality $\aleph_1$ and put $\mathscr{A} := \{A \subset X: A$ or $X - A$ is countable$\}$. For any $\mathscr{A}$-set A, define $\mu(A) = 0$ or 1 according as A or $X - A$ is countable. Furthermore, let $\mathscr{K} = \mathscr{L} = \mathscr{P}(X)$. Then $(X, \mathscr{A}, \mu)$ is a probability space and $\mathscr{K}$ μ-approximates $\mathscr{A}$. In view of Ulam's theorem (see [13, Theorem 5.6]), μ cannot be extended to a measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K})) = \mathscr{P}(X)$.

(b) Let X be a set consisting of the three elements x, y, z. Put $\mathscr{K} = \mathscr{L} = \mathscr{P}(X)$. Then $\mathscr{K}$ is a semicompact δ-lattice. Furthermore, let $A := \{x, y\}$ and $\mathscr{A} := \{\emptyset, X, A, X - A\}$. If we set $\mu(\emptyset) = \mu(X - A) = 0$ and $\mu(A) = \mu(X) = 1$, then $(X, \mathscr{A}, \mu)$ is a probability space and $\mathscr{K}$ μ-approximates $\mathscr{A}$. However, the Dirac measures pertaining to the points x and y, respectively, are different (, $\mathscr{K}$-regular) measures on $\mathscr{P}(X)$ which extend μ.

We now give an example of a semicompact δ-lattice $\mathscr{K}$ which does not $\nu | \mathscr{A}$-approximate $\mathscr{A}$ where ν is a finite, $\mathscr{K}$-regular measure on $\sigma(\mathscr{A} \cup \mathscr{F}(\mathscr{K}))$. This means that the existence statements in Theorem 3.4 cannot be reversed.

3.11. EXAMPLE. Let X be an infinite Hausdorff space such that every real-valued continuous function on X is constant (cf. [8]). Then the Baire σ-algebra of X is trivial, i.e., $\mathscr{B}_0(X) = \{\emptyset, X\}$. Fix some point $x_0 \in X$ and denote by ν the Dirac measure pertaining to x_0, i.e., for any $B \subset X$, $\nu(B) = 1$ or 0 according as $x_0 \in B$ or $x_0 \notin B$. Obviously ν is $\mathscr{K}(X)$-regular. However, if μ denotes the restriction of ν onto $\mathscr{B}_0(X)$, then $\mathscr{K}(X)$ does not μ-approximate $\mathscr{B}_0(X)$, since the condition $\mu(X) = \sup\{\mu_*(K)\colon K \in \mathscr{K}(X)\}$ is violated. On the other hand, μ satisfies the following weaker condition

$$\mu(X) = \sup\{\mu^*(K)\colon K \in \mathscr{K}(X)\}. \tag{3.8}$$

According to [19], a finite Baire measure μ on an arbitrary topological space X which satisfies (3.8) is said to be *tight*. As a consequence of the following measure extension theorem, it will be shown in 3.16 that any tight Baire measure on a Hausdorff space X can be extended to a Radon measure on X.

3.12. THEOREM. *Assume that the lattices $\mathscr{K}$, $\mathscr{L}$ satisfy the following conditions:*

(a) *$\mathscr{K}$ is a δ-lattice.*

(b) *For any $K \in \mathscr{K}$, the set $\{L \in \mathscr{L}\colon K \subset L\}$ is nonempty.*

(c) *If $K \in \mathscr{K}$ and $L \in \mathscr{L}$, then $K - L \in \mathscr{K}$ and $L - K \in \mathscr{L}$.*

(d) *$\mathscr{L}$ separates $\mathscr{K}$.*

Furthermore, let μ be a finite measure defined on a sub-σ-algebra $\mathscr{A}$ of $\sigma(\mathscr{F}(\mathscr{K}))$ such that

$$\mu(X) = \sup\{\mu^*(K)\colon K \in \mathscr{K}\}; \tag{3.9}$$

$$\mu^*(K) = \inf\{\mu^*(L)\colon K \subset L \in \mathscr{L}\} \qquad \text{for any } K \in \mathscr{K}; \tag{3.10}$$

$$\mu^* \mid \mathscr{K} \text{ is } \sigma\text{-smooth at } \varnothing. \tag{3.11}$$

Then μ can be extended to a $\mathscr{K}$-regular measure on $\sigma(\mathscr{F}(K))$. The extension is unique provided that $\mathscr{A}$ separates $\mathscr{K}$.

Proof. Choose an increasing sequence (K_n) in $\mathscr{K}$ such that $\mu(X) = \sup_n \mu^*(K_n)$. As $\nu := \mu^* \mid \mathscr{K}$ satisfies the conditions (B1)–(B5) of [1], there exists, by [1, 3.7], a $\mathscr{K}$-regular measure ρ on $\sigma(\mathscr{F}(\mathscr{K}))$ such that $\rho \mid \mathscr{K} \leqslant \nu$ and $\rho(K_n) = \nu(K_n)$ for all $n \in N$. For any $\mathscr{A}$-set A, we thus obtain $\rho(A) = \sup\{\rho(K)\colon\ K \in \mathscr{K}, K \subset A\} \leqslant \sup\{\nu(K)\colon\ K \in \mathscr{K}, K \subset A\} \leqslant \mu^*(A) = \mu(A)$. On the other hand, we have $\rho(X) \geqslant \sup_n \rho(K_n) = \sup_n \nu(K_n) = \mu(X)$, hence $\rho(X) = \mu(X)$ and so $\rho \mid \mathscr{A} = \mu$. The uniqueness statement follows from 3.1. ∎

Now we shall apply 3.12 to some topological situations. Our first application is concerned with the extension of a set function to a Baire measure. Note that any finite Baire measure on a topological space X is $\mathscr{Z}(X)$-regular (see [19, Theorem 18, Part I]).

3.13. COROLLARY. *Let X be a topological space, $\mathscr{A}$ a sub-σ-algebra of $\mathscr{B}_0(X)$ and μ a finite measure on $\mathscr{A}$ such that $\mu^* \mid \mathscr{Z}(X)$ is σ-smooth from above (i.e., $\mu^*(\bigcap_n Z_n) = \inf_n \mu^*(Z_n)$ for every decreasing sequence (Z_n) in $\mathscr{Z}(X)$). Then μ can be extended to a Baire measure on X. The extension is unique if $\mathscr{A}$ separates $\mathscr{Z}(X)$.*

Proof. Put $\mathscr{K} := \mathscr{Z}(X)$ and $\mathscr{L} := \mathscr{U}(X)$. Then the conditions (a)–(d) of 3.12 are satisfied (cf. [1, 3.15]). In addition, we have $\mathscr{F}(\mathscr{K}) = \mathscr{K}$, hence $\sigma(\mathscr{F}(\mathscr{K})) = \mathscr{B}_0(X)$. Since μ trivially satisfies the conditions (3.9) and (3.11), it remains to verify (3.10). Let $K \in \mathscr{Z}(X)$ be given. Then there exist two decreasing sequences $(K_n) \subset \mathscr{Z}(X)$ and $(L_n) \subset \mathscr{U}(X)$ such that $K = \bigcap_n L_n = \bigcap_n K_n$ and $L_n \subset K_n$ for all $n \in N$. This implies $\mu^*(K) = \inf_n \mu^*(K_n) \geqslant \inf_n \mu^*(L_n) \geqslant \inf\{\mu^*(L): K \subset L \in \mathscr{U}(X)\} \geqslant \mu^*(K)$. ▌

Our next applications of 3.12 are concerned with Borel extensions.

3.14. COROLLARY. *Let X be a normal topological space, $\mathscr{A}$ a sub-σ-algebra of $\mathscr{B}(X)$ and μ a finite measure on $\mathscr{A}$ such that*

(i) $\mu^*(F) = \inf\{\mu^*(G): F \subset G \in \mathscr{G}(X)\}$ *for all* $F \in \mathscr{F}(X)$;

(ii) $\mu^* \mid \mathscr{F}(X)$ *is σ-smooth at* $\varnothing$.

Then μ can be extended to an $\mathscr{F}(X)$-regular Borel measure on X. The extension is unique if $\mathscr{A}$ separates $\mathscr{F}(X)$.

Proof. Apply 3.12 to $\mathscr{K} := \mathscr{F}(X)$ and $\mathscr{L} := \mathscr{G}(X)$. ▌

The existence part of the following corollary was proved by different methods in [7, Proposition 4.7(c)].

3.15. COROLLARY. *Let X be a Hausdorff space and let μ be a finite measure defined on a sub-σ-algebra $\mathscr{A}$ of $\mathscr{B}(X)$. If μ satisfies the following two conditions*

(i) $\mu(X) = \sup\{\mu^*(K): K \in \mathscr{K}(X)\}$,

(ii) $\mu^*(K) = \inf\{\mu^*(G): K \subset G \in \mathscr{G}(X)\}$ *for all* $K \in \mathscr{K}(X)$,

then μ can be extended to a Radon measure on X. The extension is unique if $\mathscr{A}$ separates $\mathscr{K}(X)$.

Proof. Our claim follows from 3.12 with $\mathscr{K} := \mathscr{K}(X)$ and $\mathscr{L} := \mathscr{G}(X)$. Note that in this case condition (3.11) is automatically satisfied. ▌

In the special case $\mathscr{A} = \mathscr{B}_0(X)$ we obtain from 3.15

3.16. COROLLARY. *Every finite tight Baire measure on a Hausdorff space X can be extended to a Radon measure on X. The extension is unique if X is* *completely Hausdorff (i.e., the continuous real-valued functions separate the points of X).*

Proof. Condition (ii) of the preceding corollary is satisfied since we have $\mu(B)=\inf\{\mu(U): B\subset U\in\mathscr{U}(X)\}$, $B\in\mathscr{B}_0(X)$, for every finite Baire measure μ on X. ∎

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